

THE PICARD RANK OF AN ENRIQUES SURFACE

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ABSTRACT. In this note, we use crystalline methods and the Tate-conjecture to give a short proof that the Picard rank of an Enriques surface is equal to its second Betti number.

1. INTRODUCTION

Enriques surfaces are one of the four classes of minimal, smooth, and proper surfaces of Kodaira dimension zero. The following fundamental result relates the Picard rank ρ to the second Betti number b_2 of these surfaces.

Theorem 1.1 (Bombieri–Mumford [BM76]). *Let X be an Enriques surface over an algebraically closed field k . Then, $\rho(X) = b_2(X) = 10$.*

Using this result, it is not difficult to show that the Néron–Severi lattice of an Enriques surface is even, unimodular, of signature $(1, 9)$, and of discriminant -1 , see [II79, Corollaire II.7.3.7]. Thus, it is isometric to $U \perp E_8$ by lattice theory, see [CDL, Chapter I.5]. In particular, there exist non-zero isotropic vectors, which implies that every Enriques surface carries a genus-one fibration. Moreover, this result is also essential for the analysis of linear systems [Co85], projective models [Co83], [Li15], automorphism groups [BP83], and moduli spaces [GH16] of these surfaces.

If $k = \mathbb{C}$, then Theorem 1.1 is an easy consequence of $H^2(\mathcal{O}_X) = 0$ and the Lefschetz theorem on $(1, 1)$ classes. On the other hand, the known proofs of this result if $\text{char}(k) > 0$ are rather delicate and complicated.

- (1) The first proof is due to Bombieri and Mumford [BM76], where they first establish with some effort the existence of a genus-one fibration $f : X \rightarrow \mathbb{P}^1$. Using this, they determine $\rho(X)$ via passing to the Jacobian surface $J(X) \rightarrow \mathbb{P}^1$ of f , which is a rational surface, and thus, satisfies $\rho = b_2$.
- (2) Another proof is due to Lang [La83], who first establishes lifting of X to characteristic zero for some classes of Enriques surfaces and then, he uses the result in characteristic zero and specialization arguments. In the remaining cases, where lifting was unclear, he proves that X is unirational, and then, uses results of Shioda to conclude.

In this note, we give a conceptual proof of Theorem 1.1 that neither makes heavy use of special properties of Enriques surfaces, nor relies on case-by-case

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analyses. The idea of our proof is similar to the easy proof over the complex numbers: we merely use that the Witt-vector cohomology group $H^2(W\mathcal{O}_X)$ is torsion (note that $H^2(\mathcal{O}_X)$ may be non-zero in positive characteristic), as well as the Tate-conjecture for Enriques surfaces over finite fields, which is an arithmetic analog of the Lefschetz theorem on $(1, 1)$ classes. We refer to Remark 2.8 for details.

This note is organized as follows:

In Section 2, we give a short proof of Theorem 1.1 assuming the Tate-conjecture for Enriques surfaces over finite fields.

In order to obtain an unconditional proof, we establish in Section 3 the Tate-conjecture for Enriques surfaces over finite fields, using as little special properties of these surfaces as possible.

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2. A SHORT PROOF ASSUMING THE TATE-CONJECTURE

In this section, we first recall the definition of Enriques surfaces, as well as a couple of their elementary properties. Then, we reduce Theorem 1.1 to the case of finite fields, and finally, give a short proof of Theorem 1.1 assuming the Tate-conjecture for Enriques surfaces over finite fields.

2.1. Enriques surfaces. Let X be a smooth and proper variety (geometrically integral scheme of finite type) over a field k . We denote numerical equivalence of divisors on X by \equiv and define the i .th Betti number b_i of X to be the \mathbb{Q}_ℓ -dimension of $H_{\text{ét}}^i(X, \mathbb{Q}_\ell)$, where ℓ is a prime different from $\text{char}(k)$. For a fixed algebraic closure \bar{k} of k , we set $\bar{X} := X \times_{\text{Spec } k} \text{Spec } \bar{k}$.

Definition 2.1. A smooth and proper surface X over an algebraically closed field k is called an *Enriques surface* if

$$\omega_X \equiv \mathcal{O}_X \quad \text{and} \quad b_2(X) = 10.$$

Moreover, if k is an arbitrary field, then a smooth and proper variety X over k is called an Enriques surface if \bar{X} is an Enriques surface over \bar{k} .

From the table in the introduction of [BM77], we obtain the following equalities and bounds on the cohomology of Enriques surfaces

$$(1) \quad b_1(X) = 0, \quad b_2(X) = 10, \quad \text{and} \quad h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) \leq 1.$$

This is actually everything needed to prove Theorem 2.7 below. We remark that Enriques surfaces with $h^2(\mathcal{O}_X) \neq 0$ do exist in characteristic 2, see [BM76].

2.2. Slope one and reduction to the case of finite fields. Let $W = W(k)$ be the Witt ring of a perfect field k and let K be the field of fractions of W . Let X be a smooth and proper variety over k . Then, $b_i(X)$ is equal to the rank of the W -module $H_{\text{cris}}^i(X/W)$. The following is a straight forward generalization of [II79, Proposition II.7.3.2].

Proposition 2.2. *Let X be a smooth and projective variety over an algebraically closed field k of positive characteristic that satisfies*

$$\frac{1}{2}b_1(X) = h^1(X, \mathcal{O}_X) - h^2(X, \mathcal{O}_X).$$

Then, the F -isocrystal $H_{\text{cris}}^2(X/W) \otimes_W K$ is of slope one and

$$H^2(X, W\mathcal{O}_X) = H^2(X, W\mathcal{O}_X)_{\text{tors}} = H^2(X, W\mathcal{O}_X)_{V-\text{tors}},$$

where tors denotes torsion as W -module and $V - \text{tors}$ denotes V -torsion.

PROOF. By [II79, Remarque II.6.4], the V -torsion $H_{V-\text{tors}}^2$ of $H^2(W\mathcal{O}_X)$ is isomorphic to $\text{DM}(\text{Pic}_{X/k}^0/\text{Pic}_{X/k,\text{red}}^0)$, where $M(-)$ denotes the contravariant Dieudonné module and $D(-) = \text{Hom}_W(-, K/W)$. Thus, by Dieudonné theory, the k -dimension of $H_{V-\text{tors}}^2/VH_{V-\text{tors}}^2$ is equal to the dimension of the Zariski tangent space of $\text{Pic}_{X/k}^0/\text{Pic}_{X/k,\text{red}}^0$, which is equal to $h^1(\mathcal{O}_X) - \frac{1}{2}b_1(X)$. Thus, in the exact sequence

$$\dots \rightarrow H^1(\mathcal{O}_X) \rightarrow H^2(W\mathcal{O}_X) \xrightarrow{V} H^2(W\mathcal{O}_X) \xrightarrow{\alpha} H^2(\mathcal{O}_X) \rightarrow \dots,$$

the restriction $\alpha|_{H_{V-\text{tors}}^2} : H_{V-\text{tors}}^2 \rightarrow H^2(\mathcal{O}_X)$ is surjective by our assumptions.

Next, we set $L := H^2(W\mathcal{O}_X)/H_{V-\text{tors}}^2$ and denote the map induced by V on L again by V . Using the snake lemma, we conclude $L/VL = 0$. As explained in the proof of [II79, Proposition II.7.3.2], L is V -adically separated, which implies $L = 0$. Thus, $H^2(W\mathcal{O}_X) = H_{V-\text{tors}}^2$ and this W -module is torsion.

Since the slope spectral sequence of X degenerates up to torsion [II79, Théorème II.3.2], we conclude

$$0 = H^2(W\mathcal{O}_X) \otimes_W K = (H_{\text{cris}}^2(X/W) \otimes_W K)_{[0,1[}.$$

Since X is projective over k , the hard Lefschetz theorem (see [II75] or the discussion in [II79, Section II.5.B]) implies that also the part of slope $]1, 2]$ is zero. Thus, $H_{\text{cris}}^2(X/W) \otimes_W K$ is of slope one. \square

Proposition 2.3 (Ekedahl–Hyland–Shepherd-Barron). *Let $f : \mathcal{X} \rightarrow S$ be a smooth and projective morphism such that S is Noetherian, $f_*\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_S$, and such that*

$$\frac{1}{2}b_1(\mathcal{X}_{\bar{s}}) = h^1(\mathcal{O}_{\mathcal{X}_{\bar{s}}}) - h^2(\mathcal{O}_{\mathcal{X}_{\bar{s}}})$$

for every geometric point $\bar{s} \rightarrow S$. Then, the geometric Picard rank in this family is locally constant.

PROOF. This is a special case of [EHSB, Proposition 4.2]. \square

Corollary 2.4. *In order to prove Theorem 1.1, it suffices to establish it for Enriques surfaces that can be defined over finite fields.*

PROOF. Let X be an Enriques surface over an algebraically closed field k . Then, there exists a sub- \mathbb{Z} -algebra R of k that is of finite type over \mathbb{Z} and a smooth and projective morphism $\mathcal{X} \rightarrow S := \text{Spec } R$ with $\mathcal{X} \times_S \text{Spec } k \cong X$. Moreover, if $s \in S$ is a closed point, then the residue field $\kappa(s)$ is a finite field. In particular,

the geometric fiber $\mathcal{X}_{\bar{s}}$ is an Enriques surface over $\overline{\kappa(s)}$ and we have $\rho(\mathcal{X}_{\bar{s}}) = b_2(\mathcal{X}_{\bar{s}}) = 10$ by assumption. Using Proposition 2.3, the assertion follows. \square

2.3. The Tate-conjecture (for divisors over finite fields). Let X be a smooth and proper variety of dimension d over a finite field \mathbb{F}_q , let $N_r(X)$ to be the number of \mathbb{F}_{q^r} -rational points of X , and let

$$Z(X, t) := \exp \left(\sum_{r=1}^{\infty} N_r(X) \frac{t^r}{r} \right) = \frac{P_1(t) \cdot P_3(t) \cdots P_{2d-1}(t)}{P_0(t) \cdot P_2(t) \cdots P_{2d}(t)}$$

be the zeta function of X as in [De74]. By loc.cit., there exist $\alpha_i \in \overline{\mathbb{Q}}$ such that

$$(2) \quad P_2(t) = \prod_{i=1}^{b_2(X)} (1 - \alpha_i t)$$

and such that for every embedding of fields $\mathbb{Q}(\alpha_i) \hookrightarrow \mathbb{C}$, we have $|\alpha_i| = q$. Conjecturally, these α_i determine the Picard rank of X :

Conjecture 2.5 (Tate [Ta65]). For a smooth and proper variety X over \mathbb{F}_q , the Picard rank $\rho(X)$ is equal to the multiplicity of the factor $(1 - qt)$ in $P_2(t)$.

Although there exist more general versions of this conjecture (see [Ta94], for example), this version is sufficient for our purposes. The following lemma is crucial for our discussion.

Lemma 2.6. *Let X be a smooth and proper variety over \mathbb{F}_q . If X satisfies Conjecture 2.5 and if $H_{\text{cris}}^2(\overline{X}/W) \otimes_W K$ is of slope one, then $\rho(\overline{X}) = b_2(\overline{X})$.*

PROOF. After possibly replacing \mathbb{F}_q by a finite extension, there exists a K -basis $\{e_i\}$ of $H_{\text{cris}}^2(X/W) \otimes_W K$ such that Frobenius acts as $F(e_i) = p \cdot e_i$ for all i . If $q = p^r$, then $P_2(t)$ in Equation (2) is equal to the determinant of $(\text{id} - (F^r)^*t)$ on $H_{\text{cris}}^2(X/W) \otimes_W K$, and we conclude $P_2(t) = (1 - qt)^{b_2(X)}$. Thus, the assertion follows from Conjecture 2.5. \square

Theorem 2.7. *If Conjecture 2.5 holds for Enriques surfaces over finite fields, then Theorem 1.1 holds true.*

PROOF. By Corollary 2.4, it suffices to establish Theorem 1.1 for Enriques surfaces that can be defined over finite fields. In this special case, the claim follows from Conjecture 2.5 by Proposition 2.2 and Lemma 2.6. \square

Remarks 2.8.

- (1) In order to establish Conjecture 2.5 for a smooth and proper variety X over \mathbb{F}_q , it suffices to establish it for $X \times_{\text{Spec } \mathbb{F}_q} \text{Spec } \mathbb{F}_{q^n}$ for some $n \geq 1$. Thus, conversely, Theorem 1.1 for Enriques surfaces over $\overline{\mathbb{F}_p}$ implies Conjecture 2.5 for Enriques surfaces over finite fields.

- (2) Our approach is close to the classical proof over the complex numbers sketched in the introduction. We mention the following analogies.

\mathbb{C}	$\overline{\mathbb{F}}_p$
$H^2(\mathcal{O}_X) = 0$	$H^2(W\mathcal{O}_X)$ is W -torsion
$H^{1,1}(X) = H_{\text{dR}}^2(X, \mathbb{C})$	$H_{\text{cris}}^2(X/W) \otimes_W K$ is of slope one
Lefschetz theorem on $(1, 1)$ classes	Tate conjecture for divisors

3. THE TATE–CONJECTURE FOR ENRIQUES SURFACES

So far, we established Theorem 1.1 assuming the Tate conjecture for divisors for Enriques surfaces over finite fields. At the moment, it is not clear, when this conjecture will be established in full generality, which is why we give in this section a proof of it for Enriques surfaces to obtain an unconditional proof of Theorem 1.1.

3.1. The K3-like cover. For a projective variety X over a field k , we denote by $\text{Pic}_{X/k}^\tau$ the open subgroup scheme of $\text{Pic}_{X/k}$ that parametrizes divisor classes that are numerically equivalent to zero.

Theorem 3.1 (Bombieri–Mumford [BM76, Theorem 2]). *If X is an Enriques surface over a field k , then $\text{Pic}_{X/k}^\tau$ is a finite group scheme of length 2 over k .*

We denote by $-^D := \mathcal{H}om(-, \mathbb{G}_m)$ Cartier duality for finite, flat, and commutative group schemes. Then, Theorem 3.1 and [Ra70, Proposition (6.2.1)] (see also [BM76, Section 3] for a treatment already adapted to Enriques surfaces), show that, given an Enriques surface X over k , the natural inclusion $\text{Pic}_{X/k}^\tau \rightarrow \text{Pic}_{X/k}$ gives rise to a non-trivial torsor

$$\pi : \tilde{X} \rightarrow X$$

under $(\text{Pic}_{X/k}^\tau)^D$. In particular, π is a finite and flat morphism of degree 2. Moreover, if $\text{char}(k) \neq 2$, then π is étale and \tilde{X} is a smooth surface. In any case, \tilde{X} is called the *K3-like* cover of X . The following result is a special case of [BI82, Theorem 2], see also [CDL, Chapter I.3].

Theorem 3.2 (Blass). *If X is an Enriques surface over an algebraically closed field k , then \tilde{X} is birationally equivalent to a K3 surface or to \mathbb{P}^2 .*

PROOF. Using the cohomological invariants in Equation (1) of X , it follows that \tilde{X} is an integral Gorenstein surface with $\omega_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}$ and $\chi(\mathcal{O}_{\tilde{X}}) = 2$. Let $f : Y \rightarrow \tilde{X}$ be the minimal resolution of singularities of the normalization of \tilde{X} .

CASE 1. Assume that \tilde{X} is normal with at worst rational singularities. Being Gorenstein, \tilde{X} has at worst rational double point singularities. We conclude $\omega_Y \cong f^*\omega_{\tilde{X}} \cong \mathcal{O}_Y$ and $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_{\tilde{X}}) = 2$, which identifies Y as a K3 surface.

CASE 2. If \tilde{X} is non-normal or normal with non-rational singularities, then it is easy to see that $h^0(\omega_Y^{\otimes n}) = 0$ for all $n \geq 1$. Thus, Y is of Kodaira dimension $-\infty$. Since \tilde{X} is not smooth, we have $\text{char}(k) = 2$ and π is purely inseparable. This implies $b_1(Y) = b_1(X) = 0$ and thus, Y is a rational surface, i.e., birationally equivalent to \mathbb{P}^2 . (We refer to [BI82, Theorem 2] for details.) \square

3.2. The Tate-conjecture for Enriques surfaces over finite fields.

Theorem 3.3. *Enriques surfaces over finite fields satisfy Conjecture 2.5.*

PROOF. In order to establish Conjecture 2.5 for a smooth and proper variety X over \mathbb{F}_q , it suffices to establish it for $X \times_{\mathrm{Spec} \mathbb{F}_q} \mathrm{Spec} \mathbb{F}_{q^n}$ for some $n \geq 1$. By [Ta94, Proposition (4.3)] and [Ta94, Theorem (5.2)], we have the following implications and equivalences: First, if $Y \dashrightarrow X$ is a dominant and rational map between smooth and proper varieties over \mathbb{F}_q and Y satisfies Conjecture 2.5, then so does X . Second, if Y and Y' are a smooth, proper, and birationally equivalent varieties over \mathbb{F}_q , then Conjecture 2.5 holds for Y if and only if it holds for Y' .

Now, let X be an Enriques surface over \mathbb{F}_q , let $\tilde{X} \rightarrow X$ be the K3-like cover, and let $Y \rightarrow \tilde{X}$ be a resolution of singularities. After possibly replacing \mathbb{F}_q by a finite extension, Y is birationally equivalent to a K3 surface or to \mathbb{P}^2 by Theorem 3.2. For \mathbb{P}^2 , Conjecture 2.5 is trivial, and for K3 surfaces, it is established in [Ch13], [KMP15], [MP15], [Mau12], [Ny83], and [NO85]. By the above remarks and reduction steps, this implies Conjecture 2.5 for X . \square

Combining Theorem 2.7 and Theorem 3.3, we obtain Theorem 1.1.

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